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Best approximation in polyhedral Banach spaces

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Abstract

In the present paper, we study conditions under which the metric projection of a polyhedral Banach space X onto a closed subspace is Hausdorff lower or upper semicontinuous. For example, we prove that if X satisfies $(*)$ (a geometric property stronger than polyhedrality) and $Y \subset X$ is any proximinal subspace, then the metric projection P_Y is Hausdorff continuous and Y is strongly proximinal (i.e., if $\{y_n\} \subset Y, x \in X$ and $\|y_n - x\| \to \operatorname{dist}(x, Y)$, then $\operatorname{dist}(y_n, P_Y(x)) \to 0$).

One of the main results of a different nature is the following: if X satisfies $(*)$ and $Y \subset X$ is a closed subspace of finite codimension, then the following conditions are equivalent: (a) Y is strongly proximinal; (b) Y is proximinal; (c) each element of $Y^\perp$ attains its norm. Moreover, in this case the quotient X/Y is polyhedral.

The final part of the paper contains examples illustrating the importance of some hypotheses in our main results.

Keywords: Polyhedral Banach space; Metric projection; Proximinal subspace

0. Introduction

The present paper deals with problems related to best approximation in polyhedral Banach spaces by elements of closed subspaces. It is based on three unpublished manuscripts: [7] from

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1998 and [8] from 2003 by the first two authors, and [21] from 2005 by the third author. Especially the first two preprints have circulated among interested experts and have been cited in several published articles: [12,11,13,3]. The aim of this paper is to present in a unified way the results of the three unpublished manuscripts and to extend some of these results.

Recall that a closed subspace Y of a Banach space X is called *proximinal* if $P_Y(x)$, the set of the best approximants to x in Y, is nonempty for each $x \in X$. If, moreover, for every $x \in X$ the following implication holds

$$\{y_n\}_{n\in\mathbb{N}} \subset Y, \quad \|x - y_n\| \to d(x, Y) \Longrightarrow d(y_n, P_Y(x)) \to 0, \tag{1}$$

we say that Y is *strongly proximinal*. A *relatively strongly proximinal subspace* is defined in the same way with the only change that the implication (1) holds for every $x \in X$ with $P_Y(x) \neq \emptyset$ (see Definition 2.1).

Let us briefly describe the main results of this paper. Section 1 contains notations concerning Banach spaces, followed by definitions and preliminary facts on polyhedral Banach spaces. In Section 2 we collect preliminaries concerning metric projections onto closed subspaces.

Sections 3 and 4 deal with (semi)continuity properties of the metric projection P_Y under polyhedrality-type assumptions on X. Recall that a real Banach space X is *polyhedral* (see Klee [15]) if the unit ball of any of its finite-dimensional subspaces is a polytope. We consider two properties, $(*)$ and (Δ), defined as follows.

- A set $\mathcal{B} \subset S_{X^*}$ is a *boundary* for X if for each $x \in X$ there exists $f \in \mathcal{B}$ with $f(x) = \|x\|$.
- X satisfies $(*)$ if there exists a boundary $\mathcal{B} \subset S_{X^*}$ such that $f(x) < 1$ whenever $x \in S_X$ and f is a w^*-accumulation point of $\mathcal{B}$.
- X satisfies (Δ) if there exists a boundary $\mathcal{B} \subset S_{X^*}$ such that the set $\{f \in \mathcal{B} : f(x) = 1\}$ is finite for each $x \in S_X$.

One always has the implication

$$(*) \Longrightarrow \text{polyhedral with } (\Delta),$$

but not the reverse one [9]. Every closed subspace of any $c_0(\Gamma)$ space satisfies $(*)$ (see, e.g., [9]).

Our results on metric projections in polyhedral spaces extend all known results on this topic (even those from [8]) in the following directions:

- Y is not assumed to be finite codimensional,
- Y is not assumed to be proximinal,
- the assumptions are partially relaxed from the property $(*)$ to polyhedrality with (Δ).

They are summarized in the following theorem. (Semicontinuity notions of multivalued mappings are defined in Definition 2.2.)

Theorem 0.1. *Let Y be a closed subspace of a real Banach space X.*

(a) *If X is polyhedral with (Δ), then P_Y is Hausdorff lower semicontinuous on its effective domain* $\operatorname{dom} P_Y = \{x : P_Y(x) \neq \emptyset\}$ *(Theorem 3.7). In particular, P_Y restricted to its effective domain admits a continuous selection by Michael's selection theorem.*
(b) *If X is polyhedral with (Δ), P_Y is not necessarily Hausdorff upper semicontinuous on its effective domain, even when Y is proximinal with a finite codimension (Example 6.1).*
(c) *If X satisfies $(*)$, then P_Y is Hausdorff continuous on its effective domain, and Y is relatively strongly proximinal (Theorems 4.3 and 5.1).*

The following particular cases of Theorem 0.1 have been already known for proximinal Y of finite codimension. In [11], the continuous selection part of (a) has been derived from [7], (a) for separable X has been proved in [8], and (a) for arbitrary X has been proved in [13], all three under a stronger assumption that X satisfies $(*)$. In [12], (c) has been proved in the case when X is a subspace of c_0. The Hausdorff upper semicontinuity part of (c) has been observed in [11], using [7] and [11, Theorem 3.4]; also our proof (for general Y) uses methods from [7].

In Section 5, we consider a closed subspace Y of finite codimension in X, and the following properties:

(A) Y is strongly proximinal;
(B) Y is proximinal;
(C) each element of $Y^\perp$ attains its norm (equivalently, each closed hyperplane containing Y is proximinal);
(D) the quotient space X/Y is polyhedral (equivalently, $Y^\perp$ is polyhedral).

The following implications are easy.

(A) $\Rightarrow$ [(B) and P_Y is Hausdorff upper semicontinuous].
(B) $\Rightarrow$ (C) (Observation 5.2).
[(C) and (D)] $\Rightarrow$ (B) (see our Lemma 5.3 or [13, Theorem 1.1(a)]).

The following Theorem 0.2 summarizes our results in this direction.

Theorem 0.2. *Let Y be a closed subspace of finite codimension in a real Banach space X.*

(a) (A) $\Leftrightarrow$ [(B) *and P_Y is Hausdorff upper semicontinuous] (Theorem* 5.1*).*
(b) *For X polyhedral with (Δ), we have* (B) $\Leftrightarrow$ $\big[$(C) *and* (D)$\big]$ *(Observation* 5.2 *and Theorem* 5.4*).*
(c) *For X satisfying $(*)$, we have* (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C) *(Theorem* 5.8*).*
(d) *For X polyhedral with (Δ), we have* (B) $\not\Rightarrow$ (A) *(Example* 6.1*).*
(e) *For X polyhedral, we have* (C) $\not\Rightarrow$ (B); (C) $\not\Rightarrow$ (D); (B) $\not\Rightarrow$ (D) *(Examples* 7.3 *and* 8.1*).*

The implication "$\Leftarrow$" in Theorem 0.2(a), which holds without any assumption on (co)dimension of Y, seems to be new. In Theorem 0.2(c), the implication (C) $\Rightarrow$ (B) has been proved in [7] (in [10] for subspaces of c_0), while the implication (B) $\Rightarrow$ (A), which follows from Theorem 0.1(c), has been proved in [13], as already remarked after Theorem 0.1. The equivalence (A) $\Leftrightarrow$ (B) for subspaces of c_0 is contained in [12]. The fact that (C) $\not\Rightarrow$ (B) in general Banach spaces has been shown in [17] for $X = c$ (the Banach space of all convergent sequences) which is known to be non-polyhedral.

1. Preliminaries on polyhedral Banach spaces

Throughout the paper, X denotes a real Banach space such that $\dim X \geq 2$, with closed unit ball B_X, open unit ball B_X^0 and unit sphere S_X, and X^* is the dual of X. The set of all nonempty bounded closed convex subsets of X is denoted by $\mathcal{BCC}(X)$, and $[x, y] = \operatorname{conv}\{x, y\}$ is the closed segment with endpoints x and y. We shall use the following further notations.

By $\operatorname{ext} C$ we denote the *set of the extreme points* of a convex set C. By $\operatorname{ri} C$ we mean the *relative interior* of C in the sense of convex analysis, that is, the relative interior of C in its affine hull $\operatorname{aff} C$.

For $x \in S_X$, $D(x)$ is the image of x by the (multivalued) *duality mapping*, i.e.

$$D(x) = D_X(x) = \{f \in S_{X^*} : f(x) = 1\}.$$

Observe that $\operatorname{ext} D(x) = D(x) \cap \operatorname{ext} B_{X^*}$ by the Krein–Milman theorem.

If A is a set in X^*, then A' denotes the set of all w^*-*accumulation points* (called also w^*-limit points or w^*-cluster points) of A:

$$A' = \{f \in X^* : f \in \overline{A \setminus \{f\}}^{w^*}\}.$$

Recall also that a set $\mathcal{B} \subset B_{X^*}$ is *1-norming* if

$$\|x\| = \sup_{f \in \mathcal{B}} f(x). \tag{2}$$

A *boundary* for X is a 1-norming set $\mathcal{B} \subset B_{X^*}$ such that the supremum in (2) is in fact a maximum for each $x \in X$. The set $\mathrm{ext} B_{X^*}$ is an example of a boundary.

Definition 1.1. A set $P \in \mathcal{BCC}(X)$ is a *polytope* if the intersection of P with any finite-dimensional affine set is a (finite-dimensional) polytope.
A Banach space X is said to be *polyhedral* if B_X is a polytope.

Let us recall that X is polyhedral iff each two-dimensional subspace of X is polyhedral [14]. If X is polyhedral, then the set $w^*\text{-}\exp B_{X^*}$ (of all w^*-exposed points of B_{X^*}) coincides with the set $w^*\text{-strexp} B_{X^*}$ (of all w^*-strongly exposed points of B_{X^*}); moreover, this set is a boundary which is contained in any other boundary, and for each of its elements f, the face $f^{-1}(1) \cap S_X$ has nonempty relative interior in S_X ([4]; see also [5] or [20]).

A finite-dimensional space X is polyhedral iff X^* is polyhedral. On the other hand, an infinite-dimensional dual Banach space is never polyhedral [16] (even it is not isomorphic to any polyhedral space [4]).

Fact 1.2 (*[6]*). *If P is a separable polytope in a Banach space, then* $\mathrm{aff} P$ *is closed and* $\mathrm{ri} P \neq \emptyset$.

We shall deal with the following three geometric properties, two of them already defined in Introduction.

Definition 1.3. Let X be a Banach space.
We say that X satisfies $(*)$ if there exists a boundary $\mathcal{B} \subset S_{X^*}$ such that

$$f(x) < 1 \quad \text{whenever } x \in S_X \text{ and } f \in \mathcal{B}'. \tag{3}$$

We say that X satisfies (Δ) if there exists a boundary $\mathcal{B} \subset S_{X^*}$ such that

$$D(x) \cap \mathcal{B} \text{ is finite for each } x \in S_X. \tag{4}$$

We say that X is *(QP)* ("quasi-polyhedral" [1]) if each $x \in S_X$ has a neighborhood V such that $[x, y] \subset S_X$ whenever $y \in V \cap S_X$.

Remark 1.4. (a) The following properties are equivalent:

(i) X is a space with $(*)$;
(ii) there exists a 1-norming set $\mathcal{B} \subset S_{X^*}$ satisfying (3);
(iii) the set $\mathcal{B} = \mathrm{ext} B_{X^*}$ satisfies (3);
(iv) X is polyhedral and the set $\mathcal{B} = w^*\text{-}\exp B_{X^*}$ satisfies (3).

(Sketch of proof. If (ii) holds, then the set $\mathcal{B}_1 = \overline{\mathcal{B}}^{w^*} \cap S_{X^*}$ is easily seen to be a boundary such that $\mathcal{B}_1' = \mathcal{B}'$; thus (ii) is equivalent to (i). To see that any of (iii) and (iv) is equivalent to (i), first observe that the sets $\mathrm{ext} B_{X^*}$ and, for polyhedral X, $w^*\text{-}\exp B_{X^*}$ are boundaries; on the other hand, if $\mathcal{B}$ is a boundary then $B_{X^*} = \overline{\mathrm{conv}}^{w^*} \mathcal{B}$ and hence, by Milman's "converse" to the Krein–Milman theorem, $\overline{\mathcal{B}}^{w^*} \supset \mathrm{ext} B_{X^*}$ which implies that $(w^*\text{-}\exp B_{X^*})' \subset (\mathrm{ext} B_{X^*})' \subset \mathcal{B}'$.)

(b) Using Lemma 1.5 below, it is easy to see that the following properties are equivalent:
(i) X is a polyhedral space with (Δ);
(ii) X is polyhedral and the set $\mathcal{B} = \mathrm{ext} B_{X^*}$ satisfies (4);
(iii) X is polyhedral and the set $\mathcal{B} = w^*\text{-}\exp B_{X^*}$ satisfies (4).
(Sketch of proof. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are obvious [recall that $w^*\text{-}\exp B_{X^*}$ is a boundary for polyhedral X]. If (i) holds, then Lemma 1.5 implies that $D(x) = \mathrm{conv}[D(x) \cap \mathcal{B}]$ is a finite-dimensional polytope for each $x \in S_X$; hence the set $\mathrm{ext} D(x) = D(x) \cap \mathrm{ext} B_{X^*}$ is finite.)

Lemma 1.5. *Let X be a polyhedral Banach space, $\mathcal{B} \subset S_{X^*}$ a boundary for X, $x \in S_X$. Then*

$$D(x) = \overline{\mathrm{conv}}^{w^*}[D(x) \cap \mathcal{B}].$$

In particular, $D(x) = \mathrm{conv}[D(x) \cap \mathcal{B}]$ whenever $D(x) \cap \mathcal{B}$ is finite.

Proof. Denote $B_0 = D(x) \cap \mathcal{B}$. If the assertion is not true, there exists $f \in D(x) \setminus \overline{\mathrm{conv}}^{w^*} B_0$. By the Hahn–Banach theorem, there exists $y \in X$ such that $f(y) > \sup_{g \in B_0} g(y)$. Note that y cannot be a multiple of x since all the involved functionals have value 1 at x. Consider the two-dimensional subspace $Y = \mathrm{span}\{x, y\}$.

Since B_Y is a polygon, a part of S_Y consists of two nondegenerate line segments $[x, v_1]$ and $[x, v_2]$, where v_1, v_2 are two of the vertices of B_Y. For $i = 1, 2$, fix an arbitrary $w_i \in (x, v_i)$ and choose $g_i \in \mathcal{B}$ such that $g_i(w_i) = 1$. This implies that $[x, v_i] \subset g_i^{-1}(1)$ and hence $g_i \in B_0$. It is easy to see that $f|_Y \in D_Y(x) = [g_1|_Y, g_2|_Y]$. But then we get $f(y) \le \max\{g_1(y), g_2(y)\} \le \sup_{g \in B_0} g(y)$, a contradiction. $\square$

It is well known that the properties defined in Definition 1.3 are hereditary and, moreover, they are satisfied by any finite-dimensional polyhedral space X; for this and the following fact see, e.g., [9].

Fact 1.6. *The following implications hold:*
(a) $(*) \Rightarrow$ *(QP) with* $(\Delta) \Leftrightarrow$ *polyhedral with* (Δ)*;*
(b) *(QP)* $\Rightarrow$ *polyhedral.*

Moreover, none of the simple implications "$\Rightarrow$" can be reversed.

Observation 1.7. *A Banach space X is polyhedral with (Δ) if and only if for each $x \in S_X$ there exist a neighborhood V of x and finitely many closed halfspaces $H_1, \ldots, H_n$, each containing B_X, such that $B_X \cap V = (H_1 \cap \cdots \cap H_n) \cap V$ (that is, roughly speaking, each $x \in S_X$ has a neighborhood in which B_X coincides with a finite intersection of closed halfspaces containing B_X).*

Proof. Let X be polyhedral with (Δ). By Fact 1.6, X is (QP). It follows easily (see also [2]) that there exists a neighborhood U of x such that $D(y) \subset D(x)$ whenever $y \in U_1 := U \cap S_X$. The set $B_0 := D(x) \cap \mathcal{B}$ is finite and, by Lemma 1.5, $D(x) = \mathrm{conv} B_0$. Thus, for any $y \in U_1$, $\|y\| = 1 = \sup_{f \in D(x)} f(y) = \max_{f \in B_0} f(y)$. The open set $V := \bigcup_{\lambda > 0} \lambda U_1$ contains x and satisfies $V \cap B_X = V \cap \bigcap_{f \in B_0} H_f$ where $H_f = \{z \in X : f(x) \le 1\}$.

On the other hand, if X satisfies the condition with halfspaces, it is (QP) and hence polyhedral. Moreover, the norm-one functionals that define all involved halfspaces form a boundary $\mathcal{B}$ that satisfies (4) in Definition 1.3. $\square$

The following fact is an easy consequence of the definition of property $(*)$.

Fact 1.8. *Let X be polyhedral with $(*)$, $x \in S_X$. Then*

$$\sup\{h(x) : h \in \mathcal{B} \setminus D(x)\} < 1,$$

where $\mathcal{B}$ is any boundary satisfying (3) in Definition 1.3.

Lemma 1.9. *Let X be a polyhedral Banach space, $\mathcal{B} \subset S_{X^*}$ a boundary for X, $x, y \in X$ such that $[x, y] \cap B_X = \{x\}$. Then there exists $h \in \mathcal{B}$ such that $h(x) = 1$ and $h(y) > 1$.*

Proof. The assumptions imply that $x \in S_X$ and $y \notin B_X$. If y is a (necessarily positive) multiple of x, then any $h \in D(x) \cap \mathcal{B}$ works. Now, assume that $Z := \operatorname{span}\{x, y\}$ has dimension two. Then B_Z is a polygon. If $x \notin \operatorname{ext} B_Z$, then x is an interior point of one of the faces of B_Z. Then any $h \in D(x) \cap \mathcal{B}$ works since $\|z\| = h(z)$ whenever $z \in Z$ is sufficiently near to x. If $x \in \operatorname{ext} B_X$, then two distinct faces F_1, F_2 of B_Z meet at x. Since $\mathcal{B}$ is a boundary, there exist $h_1, h_2 \in \mathcal{B}$ such that $F_i \subset h_i^{-1}(1)$ $(i = 1, 2)$. Then $\|z\| = \max\{h_1(z), h_2(z)\}$ whenever $z \in Z$ is sufficiently near to x. It follows that, for some $i \in \{1, 2\}$, $h = h_i$ works. □

Lemma 1.10. *Let X be a polyhedral Banach space, $\mathcal{B} \subset S_{X^*}$ a boundary for X, $x_0 \in S_X$. Consider the sets*

$$B_0 = D(x_0) \cap \mathcal{B}, \qquad A = \bigcap_{h \in B_0} h^{-1}(1), \qquad F = A \cap S_X = A \cap B_X.$$

Then $A = \operatorname{aff} F$ and $x_0 \in \operatorname{ri} F$.

Proof. Obviously, the affine set A and the convex set F are closed. If $A = \{x_0\}$, we have also $F = \{x_0\}$ and the assertion is satisfied. Now, suppose $A \neq \{x_0\}$. Fix an arbitrary $x \in A \setminus \{x_0\}$ and observe that $Y := \operatorname{span}\{x_0, x\}$ has dimension two. If $x_0 \in \operatorname{ext} B_Y$ then two distinct faces of the polygon B_Y meet at x_0. Denote by C one of these two faces that does not contain x. Since $\mathcal{B}$ is a boundary for X, there exists $h \in \mathcal{B}$ such that $C \subset h^{-1}(1)$. But in this case we have $h(x_0) = 1$ and $h(x) < 1$, a contradiction with the fact that $x \in A$. Hence x_0 is an interior point of a face of B_Y.

In fact, we have proved that each line in A containing x_0 intersects F in a nondegenerate segment with x_0 in its relative interior, that is, x_0 is an algebraic interior point of F in A. A standard Baire category argument implies that $x_0 \in \operatorname{int}_A F$, which completes the proof. □

2. Preliminaries on metric projections

In what follows, Y is a closed subspace of a Banach space X, and $q: X \to X/Y$ is the corresponding quotient map. Recall that the *metric projection onto* Y is the multivalued mapping

$$P_Y: X \to 2^Y, \qquad P_Y(x) = \{y \in Y : \|x - y\| = d(x, Y)\},$$

where $d(x, Y) = \operatorname{dist}(x, Y) = \inf_{y \in Y} \|x - y\|$. We say that Y is *proximinal* if $P_Y(x) \neq \emptyset$ for each $x \in X$; and Y is *strongly proximinal* [11] if $P_Y(x) \neq \emptyset$ and $d(y_n, P_Y(x)) \to 0$ whenever $x \in X$, $\{y_n\} \subset Y$, $\|x - y_n\| \to d(x, Y)$.

The following definition weakens the notion of strong proximinality by considering only the points $x \in X$ for which $P_Y(x)$ is nonempty.

Definition 2.1. We shall say that Y is *relatively strongly proximinal* if

$$d(y_n, P_Y(x)) \to 0$$

whenever $x \in X$, $P_Y(x) \neq \emptyset$, $\{y_n\} \subset Y$, $\|x - y_n\| \to d(x, Y)$.

Let us recall basic definitions about multivalued mappings. For our purposes it suffices to remain within the framework of normed linear spaces.

Definition 2.2. Let L, M be normed linear spaces, $F: L \to 2^M$, $x_0 \in L$.

(a) F is *l.s.c.* (lower semicontinuous) at x_0 if for each open set $A \subset M$ such that $A \cap F(x_0) \neq \emptyset$ there exists a neighborhood $V \subset L$ of x_0 such that $A \cap F(x) \neq \emptyset$ whenever $x \in V$.
(b) F is *u.s.c.* (upper semicontinuous) at x_0 if for each open set $A \subset M$ such that $F(x_0) \subset A$ there exists a neighborhood $V \subset L$ of x_0 such that $F(x) \subset A$ whenever $x \in V$.
(c) F is *H-l.s.c.* (Hausdorff lower semicontinuous) at x_0 if for each $\varepsilon > 0$ there exists a neighborhood $V \subset L$ of x_0 such that $F(x_0) \subset F(x) + \varepsilon B_M$ whenever $x \in V$.
(d) F is *H-u.s.c.* (Hausdorff upper semicontinuous) at x_0 if for each $\varepsilon > 0$ there exists a neighborhood $V \subset L$ of x_0 such that $F(x) \subset F(x_0) + \varepsilon B_M$ whenever $x \in V$.
(e) Let "s.c." denote one of the four semicontinuity properties defined in (a)–(d). We say that F is *s.c. on a set* $E \subset L$ if the restriction $F|_E$ is s.c. at each point of E.
(f) The *effective domain* of F is the set $\mathrm{dom} F = \{x \in L : F(x) \neq \emptyset\}$.

It is easy to see that one always has the implications H-l.s.c. $\Rightarrow$ l.s.c., and u.s.c. $\Rightarrow$ H-u.s.c.. Moreover, F is both H-l.s.c. and H-u.s.c. at x_0 if and only if F is continuous at x_0 with respect to the Hausdorff pseudometric

$$d_H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}$$

on 2^M. (Note that d_H, restricted to the closed elements of 2^M, is a metric with values in $[0, \infty]$.)

Definition 2.3. Given a closed subspace $Y \subset X$, we define the multivalued mapping

$$R_Y: X/Y \to 2^X, \qquad R_Y(\xi) = q^{-1}(\xi) \cap B_X,$$

where $q: X \to X/Y$ is the quotient map.

Observe that $\mathrm{dom} R_Y = q(B_X)$ and this set contains $B^0_{X/Y} = q(B^0_X)$. It is easy to see that Y is proximinal if and only if $q(B_X) = B_{X/Y}$.

Appropriate versions of the following technical lemma and its corollary (Corollary 2.5) are true for bounded closed convex sets. However, for simplicity of formulation, we state them just for B_X.

Lemma 2.4. *Suppose that*

$$\xi_0, \xi \in q(B_X), \qquad \xi \neq \xi_0, \qquad x_0 \in R_Y(\xi_0), \qquad x \in R_Y(\xi),$$
$$r > 0, \quad x + \frac{r(x - x_0)}{\|x - x_0\|} \in B_X.$$

Then

$$\sup_{z_0 \in R_Y(\xi_0)} d(z_0, R_Y(\xi)) \le \frac{2\|x - x_0\|}{r}.$$

Proof. Fix an arbitrary $z_0 \in R_Y(\xi_0)$. Define $z = x + \frac{r}{\|x-x_0\|+r}(z_0 - x_0)$, and observe that $z \in q^{-1}(\xi)$. An easy calculation shows that, for $u_x = x + \frac{r(x-x_0)}{\|x-x_0\|}$, we have

$$z = \frac{\|x - x_0\|}{\|x - x_0\| + r} u_x + \frac{r}{\|x - x_0\| + r} z_0.$$

Consequently, $z \in B_X$ since $u_x, z_0 \in B_X$. It follows that $z \in R_Y(\xi)$, and hence $d(z_0, R_Y(\xi)) \leq \|z - z_0\| = \frac{\|x-x_0\|}{\|x-x_0\|+r}\|u_x - z_0\| \leq \frac{2\|x-x_0\|}{\|x-x_0\|+r} \leq \frac{2\|x-x_0\|}{r}$. □

Corollary 2.5. *The multivalued mapping R_Y is locally Lipschitz (in the Hausdorff metric) on $B^0_{X/Y}$.*

Proof. Given $\xi_0 \in B^0_{X/Y}$, fix an arbitrary $x_0 \in q^{-1}(\xi_0) \cap B^0_X$. Let $r > 0$ be such that $x_0 + 5rB_X \subset B_X$. Consider, in X/Y, arbitrary two distinct points $\xi, \eta \in \xi_0 + rB^0_{X/Y}$. There exists $x \in q^{-1}(\xi)$ such that $\|x - x_0\| < r$. Then $x \in B_X$ implies that $x \in R_Y(\xi)$. There exists $y \in q^{-1}(\eta)$ such that $\|x - y\| < 2\|\xi - \eta\|$. Since $\|\xi - \eta\| < 2r$, we have $y \in x + 4rB_X \subset x_0 + 5rB_X \subset B_X$; hence $y \in R_Y(\eta)$. Moreover, $u_x := x + \frac{r(x-y)}{\|x-y\|} \in (x_0 + rB_X) + rB_X \subset B_X$. By Lemma 2.4, $\sup_{z \in R_Y(\eta)} d(z, R_Y(\xi)) \leq \frac{2}{r}\|x - y\| \leq \frac{4}{r}\|\xi - \eta\|$. By interchanging ξ and η, we conclude that $d_H(R_Y(\xi), R_Y(\eta)) \leq \frac{4}{r}\|\xi - \eta\|$ whenever $\xi, \eta \in \xi_0 + rB^0_{X/Y}$. □

The next lemma gives a link between semicontinuity properties of the metric projection P_Y and those of R_Y. It is based on the following simple observation.

Observation 2.6. *If $x \in X$, $d(x, Y) = 1$ and $\xi = q(x)$, then*

$$R_Y(\xi) = x - P_Y(x).$$

Proof. The formula follows from the following chain of obvious equivalences.

$$\begin{aligned} z \in R_Y(\xi) \quad &\text{iff } q(z) = \xi, \|z\| \leq 1 \\ &\text{iff } x - z = y \in Y, \|x - y\| \leq 1 \\ &\text{iff } z = x - y, y \in P_Y(x). \end{aligned}$$ □

Lemma 2.7. *Let "s.c." denote one of the properties l.s.c., u.s.c., H-l.s.c., H-u.s.c. Then P_Y is s.c. on its effective domain if and only if R_Y is s.c. on the set $\Sigma = (\mathrm{dom} R_Y) \cap S_{X/Y} = q(B_X) \cap S_{X/Y}$.*

Proof. First, notice that P_Y is semi-linear with respect to Y in the sense that $P_Y(tx) = tP_Y(x)$ and $P_Y(x + y) = P_Y(x) + y$ whenever $x \in X$, $y \in Y$ and $t \in \mathbb{R}$. Moreover, the restriction $(P_Y)|_{\mathrm{dom} P_Y}$ is obviously s.c. at each point of Y. It follows easily by homogeneity that P_Y is s.c. on its effective domain if and only if P_Y is s.c. on the set

$$S = q^{-1}(S_{X/Y}) \cap \mathrm{dom} P_Y = \{x \in \mathrm{dom} P_Y : d(x, Y) = 1\}.$$

For $x \in S$, Observation 2.6 implies that $P_Y(x) = x - R_Y(q(x))$ and $q(x) \in \Sigma$. It follows that P_Y is s.c. on S whenever R_Y is s.c. on Σ.

On the other hand, the multivalued mapping $q^{-1}: X/Y \to 2^X$ is l.s.c. (since q is open) and hence admits a continuous selection σ by Michael's selection theorem. Now, for $\xi \in \Sigma$, we have $d(\sigma(\xi), Y) = \|\xi\|_{X/Y} = 1$ and $R_Y(\xi) = \sigma(\xi) - P_Y(\sigma(\xi))$ (Observation 2.6), and hence $\sigma(\xi) \in S$. It follows that R_Y is s.c. on Σ whenever P_Y is s.c. on S. □

Lemma 2.8 (*Separable Reduction*). *Assume that our multivalued mapping R_Y is not H-u.s.c. on $q(B_X)$. Then X contains a separable closed subspace X_0 such that, for $Y_0 = Y \cap X_0$, the corresponding mapping*

$$R_{Y_0}: X_0/Y_0 \to 2^{X_0}, \qquad R_{Y_0}(\eta) = q_0^{-1}(\eta) \cap B_{X_0}$$

(where $q_0: X_0 \to X_0/Y_0$ is the quotient map) is not H-u.s.c. on $q(B_{X_0})$.

Proof. Assume that R_Y is not H-u.s.c. at some $\xi_0 \in q(B_X)$. There exist $\{\xi_n\} \subset q(B_X)$, $x_n \in R_Y(\xi_n)$ and $a > 0$ such that $d(x_n, R_Y(\xi_0)) \geq a$. Fix an arbitrary $x_0 \in R_Y(\xi_0)$ and, for each $n \geq 1$, find $z_n \in q^{-1}(\xi_n)$ such that $\|z_n - x_0\| < \|\xi_n - \xi_0\| + \frac{1}{n}$. Define

$$X_0 = \overline{\operatorname{span}}\left[\{x_n\}_{n\geq 0} \cup \{z_n\}_{n\geq 1}\right].$$

The subspace $Y_0 = Y \cap X_0$ contains all the points $z_n - x_n$ $(n \geq 1)$. Put $\eta_n = q_0(z_n)$ and $\eta_0 = q_0(x_0)$, and observe that $\eta_n \to \eta_0$ since $z_n \to x_0$. For $n \geq 0$, we have $x_n \in q_0^{-1}(\eta_n) \cap B_{X_0} = R_{Y_0}(\eta_n)$. Since $R_{Y_0}(\eta_0) = (x_0 + Y_0) \cap B_{X_0} \subset (x_0 + Y) \cap B_X = R_Y(\xi_0)$, we have

$$d(x_n, R_{Y_0}(\eta_0)) \geq d(x_n, R_Y(\xi_0)) \geq a \quad (n \geq 1)$$

which shows that R_{Y_0} is not H-u.s.c. at η_0. □

3. Hausdorff lower semicontinuity of P_Y

As a starting point, we shall prove a result about lower semicontinuity (rather than Hausdorff lower semicontinuity) of P_Y (Theorem 3.5), which will be used also in Section 4. The main tool is the following proposition.

Proposition 3.1. *Let Y be a closed subspace of a Banach space X. Let $H_1, \ldots, H_n$ be closed halfspaces in X. Then the mapping $F: X^n \to 2^Y$, given by*

$$F(x_1, \ldots, x_n) = Y \cap \bigcap_{i=1}^{n} (x_i + H_i)$$

is lower semicontinuous on its effective domain.

We shall prove Proposition 3.1 in several steps.

Lemma 3.2. *Let $H_1, \ldots, H_n$ be closed halfspaces in a normed linear space X. Then the mapping $F: X^n \to 2^X$, given by*

$$F(x_1, \ldots, x_n) = \bigcap_{i=1}^{n} (x_i + H_i)$$

is lower semicontinuous on dom F.

Proof. The case of $\dim X < \infty$ was proved in [19, Proposition 5.12]. The general case easily follows. Indeed, if $H_i = \{x \in X : f_i(x) \geq t_i\}$, $L = \bigcap_{i=1}^n f_i^{-1}(0)$ and $q: X \to X/L$ is the quotient map, the sets $\widetilde{H}_i = q(H_i)$ are hyperplanes in the (finite-dimensional) space X/Y. Hence the mapping $\widetilde{F}: (X/L)^n \to 2^{X/L}$, $\widetilde{F}(\xi_1, \ldots, \xi_n) = \bigcap_{i=1}^n (\xi_i + \widetilde{H}_i)$, is lower semicontinuous on its effective domain. The rest follows from the fact that $F = q^{-1} \circ \widetilde{F} \circ Q$ where $Q(x_1, \ldots, x_n) = \big(q(x_1), \ldots, q(x_n)\big)$, since Q is continuous and q is open. □

Let us recall the following easy and well-known fact.

Fact 3.3. *Let Y be a closed subspace of a Banach space X. Then there exists a continuous retraction p of X onto Y.*

Proof. Let $q: X \to X/Y$ be the quotient map and G be a positively homogeneous continuous selection of q^{-1} (the so-called Bartle–Graves mapping). Then $p(x) = x - G(q(x))$ defines the desired retraction. □

Lemma 3.4. *Let Y be a closed subspace of a Banach space X. Let H be a closed halfspace in X that contains no translate of Y. Then $\widetilde{H} = Y \cap H$ is a closed halfspace in Y, and there exists a continuous retraction r of X onto Y such that*

$$Y \cap (x + H) = r(x) + \widetilde{H} \quad \textit{for each } x \in X.$$

Proof. Let $f \in X^* \setminus Y^\perp$ and $t \in \mathbb{R}$ be such that $H = \{x \in X : f(x) \geq t\}$. Obviously $\widetilde{H}$ is a closed halfspace in Y since f is not constant on Y. Fix $y_0 \in Y$ such that $f(y_0) = 1$. By Fact 3.3, there exists a continuous retraction p of $f^{-1}(0)$ onto $Y \cap f^{-1}(0)$. Then the mapping $r(x) = f(x)y_0 + p\big(x - f(x)y_0\big)$ is a continuous retraction onto Y such that $f\big(r(x)\big) = f(x)$ for all $x \in X$. This easily implies the assertion. □

Proof of Proposition 3.1. If $(x_1, \ldots, x_n) \in \operatorname{dom} F$ and some translate of Y belongs to H_i for some i, then necessarily $Y \subset x_i + H_i$. Hence we can (and do) suppose that Y is not parallel to any ∂H_i, the topological boundary of H_i $(i = 1, \ldots, n)$. By Lemma 3.4,

$$F(x_1, \ldots, x_n) = \bigcap_{i=1}^{n} \big(r_i(x_i) + \widetilde{H}_i\big)$$

where $r_i: X \to Y$ is a continuous retraction and $\widetilde{H}_i = Y \cap H_i$ is a closed halfspace in Y $(i = 1, \ldots, n)$. By Lemma 3.2, the mapping $Y^n \to 2^Y$, $(y_1, \ldots, y_n) \mapsto \bigcap_{i=1}^n (y_i + \widetilde{H}_i)$, is lower semicontinuous on its effective domain; hence also F is. □

Theorem 3.5. *Let X be a polyhedral Banach space with (Δ), $Y \subset X$ a closed subspace. Then the corresponding mapping R_Y is l.s.c. on its effective domain $q(B_X)$.*

Proof. We want to prove that the restriction $R_Y|_{q(B_X)}$ is l.s.c. at each $\xi_0 \in q(B_X)$. This is certainly true for $\xi_0 \in B^0_{X/Y}$ by Corollary 2.5.

Now, let $\xi_0 \in q(B_X) \cap S_{X/Y}$. Fix $x_0 \in R_Y(\xi_0)$ and an open neighborhood V of x_0. Since $x_0 \in S_X$, we can apply Observation 1.7: by taking a smaller neighborhood we can suppose that there exist finitely many closed halfspaces $H_i \subset X$ $(i = 1, \ldots, n)$ such that

$$B_X \subset \bigcap_{i=1}^{n} H_i \quad \text{and} \quad V \cap B_X = V \cap \bigcap_{i=1}^{n} H_i.$$

Observe that $x_0 \in R_Y(\xi_0) \cap V = (x_0 + Y) \cap B_X \cap V = (x_0 + Y) \cap \bigcap_{i=1}^n H_i \cap V = x_0 + \big[Y \cap \bigcap_{i=1}^n (H_i - x_0) \cap (V - x_0)\big]$. Thus $0 \in \Phi(x_0)$, where the multivalued mapping

$$\Phi(x) := Y \cap \bigcap_{i=1}^{n} (H_i - x)$$

is l.s.c. on its effective domain (Proposition 3.1). Choose $\varepsilon > 0$ and an open neighborhood W of x_0 so that $W + \varepsilon B_X \subset V$. By the lower semicontinuity of Φ, there exists an open neighborhood

U of x_0 such that

$$\|x - x_0\| < \varepsilon \quad \text{and} \quad \Phi(x) \cap (W - x_0) \neq \emptyset \quad \text{whenever } x \in U,\ \Phi(x) \neq \emptyset.$$

Notice that $q(U)$ is an open set in X/Y. For $\xi \in q(U) \cap q(B_X)$ choose $x \in q^{-1}(\xi) \cap U$ and observe that

$$\Phi(x) \supset Y \cap (B_X - x) = \big[(x + Y) \cap B_X\big] - x = R_Y(\xi) - x \neq \emptyset.$$

Consequently,

$$\begin{aligned}\emptyset &\neq \Phi(x) \cap (W - x_0) \subset \Phi(x) \cap (V - x)\\ &= \left[(x + Y) \cap \bigcap_{i=1}^{n} H_i \cap V\right] - x = \big[R_Y(\xi) \cap V\big] - x,\end{aligned}$$

which implies that $R_Y(\xi) \cap V \neq \emptyset$. The proof is complete. $\square$

The step from "l.s.c." to "H-l.s.c." is now guaranteed by the following easy consequence of Lemma 2.4.

Proposition 3.6. *Let X be (QP), $Y \subset X$ a closed subspace. If P_Y is l.s.c. on its effective domain, then P_Y is H-l.s.c. on its effective domain.*

Proof. By Lemma 2.7, we have to show that R_Y is H-l.s.c. on $E := q(B_X) \cap S_{X/Y}$ whenever it is just l.s.c. on E. Given $\xi_0 \in E$, choose an arbitrary $x_0 \in R_Y(\xi_0)$. The fact that X is (QP) easily implies that there exists $r > 0$ such that

$$x_0 + \frac{2r(x - x_0)}{\|x - x_0\|} \in B_X \quad \text{whenever } x \in S_X,\ 0 < \|x - x_0\| < r. \tag{5}$$

Let $\varepsilon \in (0, r)$ be given. Since $R_Y|_E$ is l.s.c. at ξ_0, there exists a neighborhood $U \subset S_{X/Y}$ of ξ_0 such that for each $\xi \in U \cap E$ there exists $x_\xi \in R_Y(\xi) \cap B^0(x_0, \varepsilon)$. Now, for $\xi \in U \cap E$, $\xi \neq \xi_0$, (5) implies that

$$u_{x_\xi} := x_\xi + \frac{r(x_\xi - x_0)}{\|x_\xi - x_0\|} = x_0 + \big(r + \|x_\xi - x_0\|\big)\frac{x_\xi - x_0}{\|x_\xi - x_0\|} \in B_X$$

since $r + \|x_\xi - x_0\| < 2r$ and $x_\xi \in S_X$. By Lemma 2.4, we have the estimate $\sup_{z_0 \in R_Y(\xi_0)} d(z_0, R_Y(\xi)) \leq \frac{2\|x_\xi - x_0\|}{r} < \frac{2\varepsilon}{r}$, which completes the proof. $\square$

Theorem 3.7. *Let X be a polyhedral Banach space with (Δ), $Y \subset X$ a closed subspace. Then P_Y is H-l.s.c. on its effective domain.*

Proof. By Theorem 3.5 and Lemma 2.7, P_Y is l.s.c. on its effective domain. Now, Fact 1.6(a) and Proposition 3.6 conclude the proof. $\square$

4. Hausdorff upper semicontinuity of P_Y

As we shall see in Example 6.1, property (Δ) of a polyhedral Banach space is not sufficient for Hausdorff upper semicontinuity of P_Y, even if Y is proximinal and of codimension two. In Theorem 4.2, we give a positive result under the stronger assumption that X is a Banach space with $(*)$. Let us start with the following simple

Observation 4.1. *Let M, Y be subspaces of a vector space X. If M has finite codimension in X, then $M \cap Y$ has finite codimension in Y.*

Proof. Put $N = M \cap Y$. Let Y_1 be an algebraic complement of N in Y. Then $M \cap Y_1 = (M \cap Y) \cap Y_1 = N \cap Y_1 = \{0\}$. Consequently, $\operatorname{codim}_Y N = \dim Y_1 \leq \operatorname{codim}_X M < \infty$. □

Recall that, given a closed subspace Y of X, $q: X \to X/Y$ denotes the quotient map, and $R_Y: X/Y \to 2^X$ is defined by $R_Y(\xi) = q^{-1}(\xi) \cap B_X$.

Theorem 4.2. *Let X be a polyhedral Banach space with $(*)$, $Y \subset X$ a closed subspace. Then the corresponding mapping R_Y is H-u.s.c. on its effective domain $q(B_X)$.*

Proof. By separable reduction (Lemma 2.8), we may assume that X is separable. Suppose that R_Y is not H-u.s.c. at some $\xi_0 \in q(B_X)$. There exist $\{\xi_n\} \subset q(B_X)$, $z_n \in R_Y(\xi_n)$ and $a > 0$ such that $\xi_n \to \xi_0$ and $d(z_n, R_Y(\xi_0)) > a$.

By Corollary 2.5, we must have $\xi_0 \in S_{X/Y}$. Since $R_Y(\xi_0)$ is a separable polytope, Fact 1.2 assures that $L := \operatorname{aff} R_Y(\xi_0)$ is closed and there exists $x_0 \in \operatorname{ri} R_Y(\xi_0)$ (the relative interior of $R_Y(\xi_0)$). Consider the sets

$$B_0 = D(x_0) \cap \mathcal{B}, \qquad A = \bigcap_{h \in B_0} h^{-1}(1), \qquad F = A \cap S_X = A \cap B_X.$$

By Lemma 1.10, $A = \operatorname{aff} F$ and $x_0 \in \operatorname{ri} F$. Let us denote $R_0 = R_Y(\xi_0) - x_0$, $L_0 = L - x_0$, $F_0 = F - x_0$, $A_0 = A - x_0$.

We claim that

$$L_0 = A_0 \cap Y. \tag{6}$$

To see this, notice that $R_Y(\xi_0) \subset S_X$ and $x_0 \in \operatorname{ri} F$ imply $R_Y(\xi_0) \subset A$. Then

$$F \cap (x_0 + Y) = A \cap B_X \cap (x_0 + Y) = A \cap R_Y(\xi_0) = R_Y(\xi_0),$$

and hence $A_0 \cap Y = \mathbb{R}^+ F_0 \cap Y = \mathbb{R}^+(F_0 \cap Y) = \mathbb{R}^+ R_0 = L_0$ (where $\mathbb{R}^+ E$ denotes the set of all positive multiples of the elements of E), which is (6).

Since A_0 is a subspace of finite codimension in X, by Observation 4.1 we can write

$$Y = L_0 \oplus V \tag{7}$$

where V is a finite-dimensional subspace.

By Theorem 3.5, R_Y is l.s.c. on $q(B_X)$, hence there exist points $x_n \in R_Y(\xi_n)$ such that $x_n \to x_0$. Since $z_n - x_n \in Y$, (7) implies that we can write

$$z_n = x_n + y_n + v_n \quad \text{where } y_n \in L_0, v_n \in V.$$

By passing to a subsequence, we can suppose that $v_n \to v \in V$.

We claim that $v = 0$. Indeed, if not, then $v \in Y \setminus L_0 = Y \setminus A_0$. Since $x_0 \in \operatorname{ri} F$, we must have $[x_0 + v, x_0] \cap B_X = \{x_0\}$. By Lemma 1.9, there exists $h \in B_0$ such that $h(x_0 + v) > 1$. Observe that $L_0 \subset A_0 \subset h^{-1}(0)$. Thus we have $1 < h(x_0 + v) = \lim h(x_n + v_n) = \lim h(z_n - y_n) = \lim h(z_n) \leq 1$, a contradiction which proves that $v_n \to 0$.

Since $y_n \in L_0 \subset A_0$ and $x_0 \in \operatorname{int}_A F$, the numbers

$$t_n := \max\{t \geq 0 : x_0 + t y_n \in F\} = \max\{t \geq 0 : x_0 + t y_n \in R_Y(\xi_0)\}$$

are positive and there exists $r > 0$ such that $r \leq \|t_n y_n\| \leq 2$ for each n. Moreover, $\|y_n\| = \|z_n - x_n - v_n\| \geq \|z_n - x_0\| - \|x_n - x_0\| - \|v_n\|$ and $\|y_n\| \leq 2 + \|v_n\|$. Since $\|z_n - x_0\| > a$, we can suppose that $a < \|y_n\| < 3$ for each n. Then $\frac{r}{3} < t_n < \frac{2}{a}$. Passing to a subsequence, we can suppose that $t_n \to t_0 > 0$.

We claim that $t_0 < 1$. To see this, suppose the contrary, i.e., $t_0 \geq 1$. Then $t_n' := \min\{t_n, 1\} \to 1$ and $x_0 + t_n' y_n \in R_Y(\xi_0)$. Consequently

$$\begin{aligned} a &< \|z_n - x_0 - t_n' y_n\| = \|x_n + v_n + y_n - x_0 - t_n' y_n\| \\ &\leq \|x_n - x_0\| + \|v_n\| + 3(1 - t_n') \to 0. \end{aligned}$$

This contradiction proves that $0 < t_0 < 1$.

We can suppose that $t_n < 1$ for each n. Then the definition of t_n implies that $[x_0 + t_n y_n, x_0 + y_n] \cap B_X = \{x_0 + t_n y_n\}$. By Lemma 1.9, there exist functionals $h_n \in D(x_0 + t_n y_n) \cap \mathcal{B}$ such that $h_n(x_0 + y_n) > 1$. It follows that $h_n \notin D(x_0)$. Hence, by Fact 1.8, $\sup_n h_n(x_0) =: \sigma < 1$. Then

$$h_n(y_n) = \frac{1}{t_n}\big[h_n(x_0 + t_n y_n) - h_n(x_0)\big] \geq \frac{1 - \sigma}{t_n}.$$

But then we get

$$\begin{aligned} 1 &\geq \limsup h_n(z_n) = \limsup h_n(x_n + v_n + y_n) = \limsup h_n(x_0 + y_n) \\ &= \limsup\big[h_n(x_0 + t_n y_n) + (1 - t_n) h_n(y_n)\big] \geq 1 + \limsup \frac{(1 - t_n)(1 - \sigma)}{t_n} \\ &= 1 + \frac{(1 - t_0)(1 - \sigma)}{t_0} > 1, \end{aligned}$$

a contradiction which completes the proof. □

Theorem 4.3. *Let X be a polyhedral Banach space with $(*)$, $Y \subset X$ a closed subspace. Then Y is relatively strongly proximinal and P_Y is Hausdorff continuous on its effective domain.*

Proof. By Theorem 4.2 and Lemma 2.7, $P_Y|_{\mathrm{dom} P_Y}$ is H-u.s.c. By Theorem 3.7 and Fact 1.6, it is also H-l.s.c. Finally, Y is relatively strongly proximinal by Theorem 5.1 proved in the next section. □

Corollary 4.4. *Let X satisfy $(*)$. Then every proximinal subspace of X is strongly proximinal and the corresponding metric projection is Hausdorff continuous.*

5. Proximinality of subspaces and polyhedrality of quotients

Let Y be a closed subspace of a Banach space X. Recall that $q: X \to X/Y$ denotes the quotient map, and $P_Y: X \to 2^Y$ is the metric projection onto Y. By $NA(X)$ we mean the set of all norm-attaining elements of X^*. For definitions of proximinality and strong proximinality see Section 2.

In this section, we consider the following four properties, already introduced in Introduction:

(A) Y is strongly proximinal;
(B) Y is proximinal;
(C) $Y^\perp \subset NA(X)$;
(D) X/Y is polyhedral.

In main results of this section, Y will be of finite codimension in X.

Obviously, (A) implies (B). Let us start this section by proving several relatively simple general results which hold without any polyhedrality assumption on X:

(a) for Y proximinal, (A) holds iff P_Y is H-u.s.c. (Theorem 5.1);
(b) for X/Y reflexive, (B) implies (C) (Observation 5.2(b));
(c) for X/Y finite-dimensional, [(C) and (D)] implies (B) (Lemma 5.3).

The implication "⇐" in (a) seems to be new. In its proof (proof of Theorem 5.1), it is quite convenient to use our mapping R_Y (see Definition 2.3).

Theorem 5.1. *Let Y be a closed subspace of a Banach space X. Then Y is relatively strongly proximinal if and only if the metric projection P_Y is H-u.s.c. on its effective domain. (In particular, a proximinal subspace Y is strongly proximinal if and only if P_Y is H-u.s.c.)*

Proof. The implication "⇒" follows easily from definitions. (For Y proximinal, it has been observed in [11, Lemma 4.1] or [13, p. 240].) Let us show the other implication.

Assume that Y is not relatively strongly proximinal. This means that there exist $x \in \operatorname{dom} P_Y$, $\{y_n\} \subset Y$ and $a > 0$ such that $\|x - y_n\| \to d(x, Y)$ and $d(y_n, P_Y(x)) > a$ for each n. Since obviously $x \notin Y$, by homogeneity we can (and do) suppose that $d(x, Y) = 1$. Define

$$x_n = \frac{x}{\|x - y_n\|}, \qquad z_n = x_n - \frac{y_n}{\|x - y_n\|} = \frac{x - y_n}{\|x - y_n\|},$$
$$\xi_n = q(x_n) = q(z_n), \qquad \xi = q(x).$$

Then we have: $R_Y(\xi) = x - P_Y(x)$ (Observation 2.6), $\xi \in q(B_X) \cap S_{X/Y}$, $\xi_n \in q(B_X)$ and $z_n \in q^{-1}(\xi_n) \cap B_X = R_Y(\xi_n)$ for each n; and $\xi_n \to \xi$ since $x_n \to x$. Now, since $\|x - y_n\| \to 1$, we can write

$$\begin{aligned}
\liminf_{n\to\infty} d(z_n, R_Y(\xi)) &= \liminf_{n\to\infty} d\left(\frac{y_n}{\|x - y_n\|} - x_n,\, P_Y(x) - x\right) \\
&= \liminf_{n\to\infty} d\left(\frac{y_n}{\|x - y_n\|} + (x - x_n),\, P_Y(x)\right) \\
&= \liminf_{n\to\infty} d\left(\frac{y_n}{\|x - y_n\|},\, P_Y(x)\right) \\
&= \liminf_{n\to\infty} d(y_n, P_Y(x)) \geq a.
\end{aligned}$$

It follows that $R_Y|_{q(B_X)}$ is not H-u.s.c. at ξ. By Lemma 2.7, P_Y is not H-u.s.c. on its effective domain. □

Observation 5.2. (a) *If $Y^\perp \subset NA(X)$, then X/Y is reflexive.*
(b) *If Y is proximinal and X/Y is reflexive, then $Y^\perp \subset NA(X)$.*

Proof. (a) is an immediate consequence of the James theorem. To show (b), fix an arbitrary $f \in Y^\perp = (X/Y)^*$. There exists $\xi \in S_{X/Y}$ such that $f(\xi) = \|f\|$. Since Y is proximinal, there exists $x \in R_Y(\xi) = q^{-1}(\xi) \cap S_X$. Then $f(x) = f(\xi) = \|f\|$ implies that $f \in NA(X)$. □

The following easy fact is mentioned also in [13, Theorem 1.1(a)].

Lemma 5.3. *Let X be a Banach space, $Y \subset X$ a closed subspace of finite codimension. If $Y^\perp \subset NA(X)$ and X/Y is polyhedral, then Y is proximinal.*

Proof. Since $B_{X/Y}$ is a finite-dimensional polytope, it is a convex hull of its extreme points (that are also exposed points, in this case). For $\xi \in \operatorname{ext} B_{X/Y}$, take $f \in S_{Y^\perp}$ such that $f(\xi) = 1$ and $f(\eta) < 1$ whenever $\eta \in B_{X/Y} \setminus \{\xi\}$. Since $f \in NA(X)$, there exists $x \in S_X$ with $1 = f(x) = f(q(x))$. By the choice of f, we must have $q(x) = \xi$. We have proved that $\operatorname{ext} B_{X/Y} \subset q(B_X)$. Consequently, $B_{X/Y} = \operatorname{conv}(\operatorname{ext} B_{X/Y}) \subset q(B_X)$, which implies that $q(B_X) = B_{X/Y}$. And this is equivalent to proximinality of Y. □

In the rest of this section, as well as in the following sections containing counterexamples, we consider the properties (A)–(D) in the case of a finite-codimensional subspace Y of X, under suitable assumptions on X, stronger than polyhedrality (namely, property $(*)$ or polyhedrality with (Δ)). Our main results are summarized in Theorem 0.2 (see Introduction).

See Definition 1.3 for properties $(*)$ and (Δ).

Theorem 5.4. *Let X be a polyhedral Banach space with (Δ), $Y \subset X$ a closed subspace of finite codimension. If Y is proximinal then the quotient X/Y is polyhedral.*

Proof. We have to prove that the finite-dimensional space $Y^{\perp}$ (the dual of X/Y) is polyhedral. Suppose this is not the case. Then there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathrm{ext} B_{Y^{\perp}}$ of pairwise distinct functionals. Let $\xi_n \in S_{X/Y}$ be such that $f_n(\xi_n) = 1$ ($n \geq 1$). By compactness (X/Y has finite dimension!), we can suppose that $\xi_n \to \xi_0$. By proximinality of Y and by Theorem 3.5, the mapping $R_Y(\xi) = q^{-1}(\xi) \cap B_X$ has nonempty values and is lower semicontinuous on $S_{X/Y}$; hence it admits a continuous selection (Michael's theorem). It follows that there exist points $x_n \in S_X$ such that $q(x_n) = \xi_n$ for all $n \geq 0$, and $x_n \to x_0$. Observe that $f_n \in D(x_n)$ for each $n \geq 1$.

By Fact 1.6, X is (QP); hence $D(z) \subset D(x_0)$ for each $z \in S_X$ sufficiently close to x_0 (cf. [2]). It follows that $f_n \in D(x_0)$ for each sufficiently large n. Observe that the duality mapping of X/Y satisfies $D_{X/Y}(\xi_0) = D(x_0) \cap Y^{\perp}$. For each sufficiently large n, we have

$$\begin{aligned} f_n \in D(x_0) \cap \mathrm{ext} B_{Y^{\perp}} &= D_{X/Y}(\xi_0) \cap \mathrm{ext} B_{(X/Y)^*} \\ &= \mathrm{ext} D_{X/Y}(\xi_0) = \mathrm{ext}\big(D(x_0) \cap Y^{\perp}\big). \end{aligned}$$

But this is a contradiction since the last set is finite (indeed, $D(x_0)$ is a finite-dimensional polytope by the property (Δ) and Lemma 1.5). $\square$

In the last part of this section, we shall need some finer properties of polyhedral spaces. For simplicity, we use the following notation, valid only in the current section: given a boundary $\mathcal{B} \subset S_{X^*}$, we denote

$$\Lambda = \{\lambda \in \ell_1^+(\mathcal{B}) : \|\lambda\|_1 \leq 1\}, \qquad \Lambda_1 = \{\lambda \in \ell_1^+(\mathcal{B}) : \|\lambda\|_1 = 1\},$$

where $\ell_1^+(\mathcal{B})$ is the positive cone of the Banach lattice $\ell_1(\mathcal{B})$.

Lemma 5.5. *Let X be a Banach space with $(*)$, $\mathcal{B} \subset S_{X^*}$ the corresponding boundary. Let a sequence $\{\lambda_n\} \subset \Lambda_1$ be such that the functionals*

$$f_n = \sum_{h \in \mathcal{B}} \lambda_n(h) h \quad (n \in \mathbb{N})$$

converge in the weak* topology to some $f \in S_{X^*} \cap NA(X)$. Then there exist $\lambda \in \Lambda_1$ and an increasing sequence $\{n_k\}$ of positive integers such that:

- *λ has a finite support* $\mathrm{supp}(\lambda)$,
- *$f = \sum_{h \in \mathcal{B}} \lambda(h) h$,*
- *$\|f_{n_k} - f\| \to 0$, $\|\lambda_{n_k} - \lambda\|_1 \to 0$.*

Proof. Since $\bigcup_{n \geq 1} \mathrm{supp}(\lambda_n)$ is countable, a standard diagonal method gives a subsequence of $\{\lambda_n\}$ that converges pointwise to some $\lambda \in \Lambda$; for simplicity, let us denote it again by $\{\lambda_n\}$.

Let $x_0 \in S_X$ be such that $f(x_0) = 1$. Since X has $(*)$, the set $B_0 := D(x_0) \cap \mathcal{B}$ is finite. By Fact 1.8, $\sigma := \sup_{h \in \mathcal{B} \setminus B_0} h(x_0) < 1$. Now, we have

$$f_n(x_0) = \sum_{h \in \mathcal{B}} \lambda_n(h) h(x_0) \le \sum_{h \in B_0} \lambda_n(h) + \sigma \sum_{h \in \mathcal{B} \setminus B_0} \lambda_n(h) = (1 - \sigma) \sum_{h \in B_0} \lambda_n(h) + \sigma.$$

It follows that

$$\sum_{h \in B_0} \lambda_n(h) \ge \frac{f_n(x_0) - \sigma}{1 - \sigma}.$$

Passing to limits, we obtain $\sum_{h \in B_0} \lambda(h) \ge 1$. Consequently, $\|\lambda\|_1 = 1$ and $\operatorname{supp}(\lambda) \subset B_0$. By the well-known fact that pointwise and norm convergence coincide on the unit sphere of $\ell_1(\mathcal{B})$, we get that $\|\lambda_n - \lambda\|_1 \to 0$. And this easily implies that $\|f_n - f\| \to 0$. □

As a consequence of Lemma 5.5, we get the following proposition. Notice that $S_{X^*} \cap NA(X) = D(S_X)$.

Proposition 5.6. *Let X be a Banach space with $(*)$. Let $\{f_n\} \subset D(S_X)$ be a sequence converging in the weak* topology to a functional $f \in D(S_X)$. Then $D^{-1}(f_n) \subset D^{-1}(f)$ for each sufficiently large n.*

Proof. Assume the contrary. Passing to a subsequence, we can suppose that

$$D^{-1}(f_n) \not\subset D^{-1}(f) \quad \text{for each } n.$$

By Lemma 1.5, we have $f_n, f \in \operatorname{conv}\mathcal{B}$, where $\mathcal{B} \subset S_{X^*}$ is a boundary satisfying (3) in Definition 1.3. By Lemma 5.5, passing to a further subsequence, we can suppose that f_n, f can be expressed as convex combinations

$$f_n = \sum_{h \in \mathcal{B}} \lambda_n(h) h, \qquad f = \sum_{h \in \mathcal{B}} \lambda(h) h,$$

where $\lambda_n, \lambda \in \Lambda_1$ have finite supports and $\lambda_n \to \lambda$ in $\ell_1^+(\mathcal{B})$. There exists an index n_0 such that

$$\operatorname{supp}(\lambda) \subset \operatorname{supp}(\lambda_n) \quad \text{whenever } n \ge n_0.$$

Now, let $n \ge n_0$ and $x \in D^{-1}(f_n)$. Since $1 = f_n(x) = \sum_{h \in \mathcal{B}} \lambda_n(h) h(x)$, we must have $h(x) = 1$ whenever $h \in \operatorname{supp}(\lambda_n)$. It follows that

$$f(x) = \sum_{h \in \operatorname{supp}(\lambda)} \lambda(h) h(x) = \sum_{h \in \operatorname{supp}(\lambda)} \lambda(h) = 1,$$

that is, $x \in D^{-1}(f)$. We have proved that $D^{-1}(f_n) \subset D^{-1}(f)$, which is a contradiction. □

Let us state the following theorem of independent interest, which will not be needed in the sequel.

Amir and Deutsch [1] defined the following notion: given a Banach space E, a point $x \in S_E$ is a *(QP)-point* of B_E if there exists a neighborhood U of x such that

$$[y, x] \subset S_E \quad \text{whenever } y \in U \cap S_E. \tag{8}$$

Thus the space E is (QP) if and only if each point of its unit sphere is a (QP)-point of B_E. It is easy to see (cf. [11, Section 3]) that (8) in this definition can be equivalently replaced with any of the following two conditions:

$$D_E(y) \subset D_E(x) \quad \text{whenever } y \in U \cap S_E; \tag{9}$$

$$\exists M \subset S_E \text{ dense such that: } D_E(y) \cap D_E(x) \neq \emptyset \quad \text{whenever } y \in U \cap M. \tag{10}$$

Theorem 5.7. *Let X be a Banach space with $(*)$. Then:*

(a) *weak* and norm convergence of sequences coincide in the set $D(S_X) = NA(X) \cap S_{X^*}$;*
(b) *every element of $D(S_X)$ is a (QP)-point of B_{X^*}.*

Proof. (a) and (b) follow from Lemma 5.5 and Proposition 5.6, respectively. (For (b) use (10) with $E = X^*$, $M = D(S_X)$.) $\square$

Theorem 5.8. *Let X be a Banach space with $(*)$, $Y \subset X$ a closed subspace of finite codimension. If $Y^\perp \subset NA(X)$, then the quotient X/Y is polyhedral and the subspace Y is strongly proximinal.*

Proof. By Corollary 4.4, it suffices to show that Y is proximinal. By Lemma 5.3, this will be proved once we show that X/Y is polyhedral, or equivalently, that $Y^\perp = (X/Y)^*$ is polyhedral.

If $Y^\perp$ is not polyhedral, $Y^\perp$ is not (QP). Thus there exist $f, f_n \in S_{Y^\perp}$ $(n \in \mathbb{N})$ such that $f_n \to f$ and $[f_n, f] \not\subset S_{Y^\perp}$. By Proposition 5.6, we can suppose that

$$D^{-1}(f_n) \subset D^{-1}(f) \quad (n \in \mathbb{N}).$$

Choose $x_n \in D^{-1}(f_n)$. Then $f_n(x_n) = 1$ and also $f(x_n) = 1$, which implies that $f_n, f \in D_{X/Y}(q(x_n))$. Consequently, $[f_n, f] \subset D_{X/Y}(q(x_n)) \subset S_{Y^\perp}$, which is a contradiction. $\square$

Remark 5.9. The above theorem with "proximinal" instead of "strongly proximinal" appeared in [7]. The strong proximinality part has been already observed by Godefroy and Indumathi in [11], using our Theorem 5.7(b) (proved, but not explicitly stated, in [7]) and [11, Theorem 3.4].

6. First example

The following example shows that the assumption that X satisfies $(*)$ in Corollary 4.4 cannot be substituted by the weaker assumption that X is polyhedral with (Δ).

Example 6.1. There exist a Banach space X, isomorphic to c_0, and a closed subspace $Y \subset X$ of codimension two such that:

(a) X is polyhedral with (Δ),
(b) Y is proximinal,
(c) Y is not strongly proximinal,
(d) P_Y is not H-u.s.c.

Proof. Let $\{e_n\}$ be the standard basis of c_0. For $x = \sum_{n=1}^\infty x_n e_n \in c_0$, define

$$|\!|\!|x|\!|\!| = \max\left\{\|x\|_\infty,\ \sup_{n\geq 3}\left(\frac{n}{n+1}|x_2| + \frac{2}{n+1}|x_n|\right)\right\}.$$

Clearly, $|\!|\!|\cdot|\!|\!|$ is an equivalent norm on c_0. Put $X = (c_0, |\!|\!|\cdot|\!|\!|)$.

To prove (a), fix $x \in S_X$. Find an integer $n_0 \geq 3$ such that $|x_n| < \frac{1}{8}$ whenever $n \geq n_0$. Let $y = \sum_{n=1}^{\infty} y_n e_n \in S_X$ be such that $\|y - x\|_\infty \leq \frac{1}{8}$. Then, for $n \geq n_0$, we have $|y_n| \leq \frac{1}{4}$ and

$$\frac{n}{n+1}|y_2| + \frac{2}{n+1}|y_n| \leq \frac{n}{n+1} + \frac{1}{2(n+1)} = \frac{2n+1}{2n+2} < 1.$$

It easily follows that, in a certain neighborhood of x, B_X coincides with a finite intersection of closed halfspaces. Now, (a) follows from Observation 1.7.

Consider the canonical projection $\pi_2 \colon X \to Z := \operatorname{span}\{e_1, e_2\}$, defined by $\pi_2(\sum_{n=1}^{\infty} x_n e_n) = x_1 e_1 + x_2 e_2$. The norm of X is a lattice norm, that is, $|||x||| \leq |||y|||$ whenever $x, y \in X$ are such that $|x_n| \leq |y_n|$ for each n. Let $x \in X$. Define $Y = \overline{\operatorname{span}}\{e_n\}_{n \geq 3}$ and observe that, for every $y \in Y$, we have

$$|||x - y||| \geq |||\pi_2(x - y)||| = |||\pi_2(x)||| = |||x - (x - \pi_2(x))|||.$$

Since $x - \pi_2(x) \in Y$, we have $x - \pi_2(x) \in P_Y(x)$, which proves that Y is proximinal.

By the last inequality, the quotient map $q \colon X \to X/Y$, restricted to Z, is an isometry between Z and X/Y. Thus we can consider our multivalued mapping R_Y (see Definition 2.3) as a mapping $R_Y \colon Z \to 2^X$, $R_Y(z) = (z + Y) \cap B_X$. Since Y is proximinal, $\operatorname{dom} R_Y = B_{X/Y}$. Consider the points

$$z_0 = e_1 + e_2, \qquad z_n = e_1 + \frac{n-1}{n} e_2, \qquad x_n = e_1 + \frac{n-1}{n} e_2 + e_n \quad (n \geq 3).$$

It is easy to see that $|||z_0||| = |||z_n||| = |||x_n||| = 1$. Thus we have $x_n \in R_Y(z_n)$ $(n \geq 3)$, and $z_n \to z_0$. Now, observe that every $x \in R_Y(z_0)$ is of the form $x = e_1 + e_2 + \sum_{n=3}^{\infty} t_n e_n$, where $\frac{n}{n+1} + \frac{2}{n+1}|t_n| \leq 1$. The last inequality easily implies that $|t_n| \leq \frac{1}{2}$ for every $n \geq 3$. We conclude that

$$d_{|||\cdot|||}(x_n, R_Y(z_0)) \geq d_{\|\cdot\|_\infty}(x_n, R_Y(z_0)) \geq \frac{1}{2} \quad (n \geq 3),$$

and the restriction $R_Y|_{S_Z}$ is not H-u.s.c. at z_0. By Lemma 2.7, P_Y is not H-u.s.c. By Theorem 5.1, Y is not strongly proximinal. $\square$

7. Second example

The aim of this section is to provide Example 7.3. Let us start with some preparatory facts. The criterion of polyhedrality in Proposition 7.1 is of independent interest.

For a set $A \subset X^*$, we use the following notation for its annihilators:

$$A^\top = \{x \in X : x|_A \equiv 0\}, \qquad A^\perp = \{F \in X^{**} : F|_A \equiv 0\}.$$

Proposition 7.1. *Let X be a Banach space and $\mathcal{B} \subset B_{X^*}$ a boundary for X. Assume that for each $f \in \mathcal{B}' \cap D(S_X)$ there exists a symmetric set $K \subset X^*$ such that* $\dim(K^\top) \leq 1$ *and $f + K \subset B_{X^*}$. Then X is polyhedral.*

Proof. Consider an arbitrary two-dimensional subspace Y of X. Suppose that B_Y is not a polytope. Then B_{Y^*} has infinitely many extreme points. Since $\operatorname{ext} B_{Y^*}$ is closed (hence compact), it contains pairwise distinct functionals $g_0, g_1, g_2, \ldots$ such that $g_n \to g_0$. For each $n \geq 1$, an easy application of the Krein–Milman theorem gives existence of $f_n \in \operatorname{ext} B_{X^*}$ such that $f_n|_Y = g_n$. Let f_0 be a w^*-limit point of $\{f_n\}_{n \geq 1}$. Then $f_0|_Y = g_0$ and $f_0 \in (\operatorname{ext} B_{X^*})' \subset \mathcal{B}'$, where the last inclusion follows from the Milman theorem. Moreover, for some $y \in S_Y \subset S_X$, we have $f_0(y) = g_0(y) = 1$, which implies that $f_0 \in \mathcal{B}' \cap D(S_X)$. By our assumption, there exists a symmetric set $K \subset X^*$ such that $\dim K^\top \le 1$ and $f_0 + K \subset B_{X^*}$. Since Y cannot be contained in $K^\top$, there exists $h \in K$ such that $h|_Y \ne 0$. Since $f_0 = \frac{1}{2}(f_0 + h) + \frac{1}{2}(f_0 - h)$ and $f_0 \pm h \in B_{X^*}$, we have $g_0 = \frac{1}{2}(g_0 + h|_Y) + \frac{1}{2}(g_0 - h|_Y)$ and $g_0 \pm h|_Y \in B_{Y^*}$, a contradiction with the fact that $g_0 \in \operatorname{ext} B_{Y^*}$. □

Let $I \subset \mathbb{R}$ be an interval and $\varphi: I \to \mathbb{R}$ a convex function. Recall that the *epigraph* of φ is the set

$$\operatorname{epi}(\varphi) = \{(t, s) \in I \times \mathbb{R} : s \ge f(t)\}.$$

We shall need the following simple lemma based on elementary properties of convex functions of one real variable.

Lemma 7.2. *Let $\varphi: (-\delta, \delta) \to \mathbb{R}$ be a convex function with $\varphi(0) = 0$, $\varphi(t) > 0$ for $t \in (0, \delta)$, and $\varphi'_+(0) = 0$. Then there exist points $p_n = (t_n, s_n) \in \mathbb{R}^2$ ($n \in \mathbb{N}$) such that:*

(a) $\delta > t_1 > t_2 > \cdots > 0$, $s_n > 0$ *($n \in \mathbb{N}$),* $t_n \to 0$*;*
(b) *for each n, the line $\Lambda_n = \operatorname{aff}\{p_n, p_{n+1}\}$ does not intersect the epigraph of φ;*
(c) *the slopes d_n of Λ_n ($n \in \mathbb{N}$) form a decreasing sequence.*

Sketch of proof. Take any decreasing sequence $\{\tau_n\} \subset (0, \delta)$ of smooth points of φ, such that $\tau_n \to 0$. Denoting $d_n = \frac{1}{2}\varphi'(\tau_n)$, we have $d_n \ge d_{n+1} > 0$ ($n \in \mathbb{N}$) and $d_n \to \frac{1}{2}\varphi'_+(0) = 0$ (since φ'_+ is right continuous, see [18, p. 7]). By passing to a subsequence, we can suppose that $\{d_n\}$ is decreasing.

Let Λ_n be the tangent line to the graph of $\frac{1}{2}\varphi$ at the point of abscissa τ_n, that is the line of equation

$$s = \frac{1}{2}\varphi(\tau_n) + d_n(t - \tau_n).$$

Since $\varphi(t) \ge 0$ for $t \in (-\delta, 0)$, and Λ_n supports $\operatorname{epi}(\frac{1}{2}\varphi)$ at the point of abscissa τ_n, it is easy to see that Λ_n does not intersect $\operatorname{epi}(\varphi)$. For each n, let $p_n = (t_n, s_n)$ be the point of intersection of Λ_n and Λ_{n+1}. Since $\tau_{n+1} < t_n < \tau_n$ and $\frac{1}{2}\varphi(\tau_{n+1}) < s_n < \frac{1}{2}\varphi(\tau_n)$, the points p_n have the required properties. □

Now we are ready for our second example. It shows that, in the notation of Theorem 0.2, the implications (C) ⇒ (B) and (C) ⇒ (D) fail in general polyhedral spaces. (We already know from Theorem 5.8 that they hold under the assumption that X satisfies (∗).)

Example 7.3. There exists a polyhedral Banach space E, isomorphic to c_0, and a closed subspace $Y \subset E$ of codimension two, such that $Y^\perp \subset NA(E)$, Y is not proximinal, and E/Y is not polyhedral.

The proof of Example 7.3 will be done in several steps.

First step of construction. We consider the elements of the sequence spaces c_0, ℓ_1, ℓ_∞ to be of the form $a = (a_0, a_1, a_2, \ldots)$, that is, the indexing starts with 0. Let $\{u_i\}_{i \ge 0}$ and $\{e_i\}_{i \ge 0}$ be the canonical bases of c_0 and $\ell_1 = (c_0)^*$, respectively. Define

$$K = \overline{\operatorname{conv}}\{\pm 4^{-i}(e_1 - e_i) : i \ge 2\},$$
$$V = \overline{\operatorname{conv}}^{w^*}\big[B_{\ell_1} \cup \pm(e_0 + K)\big] = \operatorname{conv}\big[B_{\ell_1} \cup \pm(e_0 + K)\big]$$

(the last equality holds since B_{ℓ_1} and K are w^*-compact and convex). Then V is the dual unit ball of an equivalent norm $\|\cdot\|$ on c_0, given by

$$\|x\| = \max x(V).$$

We define $X = (c_0, \|\cdot\|)$.

Let us define also $F_1, F_2 \in \ell_\infty$, $g \in \ell_1$ and $L \subset X^*$ by

$$F_1 = (1, 1, 1, \ldots), \qquad F_2 = (1, -1, -1, \ldots),$$
$$g = e_1 - \sum_{i \ge 2} 2^{-i} e_i,$$
$$L = \operatorname{span}\{e_0, g\}.$$

It is easy to see that $u_0 = \frac{1}{2}(F_1 + F_2)$, $K \subset \operatorname{Ker}(F_1) \cap \operatorname{Ker}(F_2)$, $u_0 \in S_X$, $e_0 \in S_{X^*}$ and $F_i \in S_{X^{**}}$ $(i = 1, 2)$. Note that $F_1(e_0) = F_2(e_0) = 1$, $F_1(g) = \frac{1}{2}$, $F_2(g) = -\frac{1}{2}$, and hence $F_1|_L$ and $F_2|_L$ are linearly independent.

Claim 1. $D_{X^*}(e_0) = [F_1, F_2]$ *(the closed segment with endpoints F_1, F_2). Consequently, $D_L(e_0) = [F_1|_L, F_2|_L]$ by the Hahn–Banach theorem.*

Proof. First, let us show that $\operatorname{Ker}(F_1) \cap \operatorname{Ker}(F_2) = \overline{\operatorname{span}}\{e_1 - e_i\}_{i \ge 2}$.
The inclusion "$\supset$" follows from the fact that $F_k(e_1 - e_i) = 0$ $(k = 1, 2,\ i \ge 1)$. The equality holds since both the left- and the right-hand side have codimension two (indeed, $\ell_1 = \overline{\operatorname{span}}\{e_1 - e_i\}_{i \ge 2} \oplus \operatorname{span}\{e_0, e_1\}$).

Now, since $F_k(e_0) = 1$ $(k = 1, 2)$, we have the inclusion $[F_1, F_2] \subset D_{X^*}(e_0)$. On the other hand, if $G \in D_{X^*}(e_0)$, then $G(e_0) = 1$ and (by symmetry of K) $G|_K \equiv 0$. Thus $G \in [\overline{\operatorname{span}}\{e_1 - e_i\}_{i \ge 2}]^\perp = [\operatorname{Ker}(F_1) \cap \operatorname{Ker}(F_2)]^\perp = \operatorname{span}\{F_1, F_2\}$. Write $G = \lambda F_1 + \mu F_2$, where $\lambda, \mu \in \mathbb{R}$. Since $1 = G(e_0) = \lambda + \mu$, we have $G = \lambda F_1 + (1 - \lambda) F_2 = (1, 2\lambda - 1, 2\lambda - 1, \ldots)$. Now, $1 \ge |G(e_1)| = |2\lambda - 1|$ implies that $\lambda \in [0, 1]$, and hence $g \in [F_1, F_2]$. $\square$

Claim 2. *If $f = ae_0 + bg \in S_L$ satisfies $b > 0$, then $F_2(f) < F_1(f) < 1$.*

Proof. The first inequality is clear: $F_1(f) = a + \frac{b}{2} > a - \frac{b}{2} = F_2(f)$. To prove the second inequality, assume the contrary, that is $F_1(f) = 1$. Since $f \in V$, we can write

$$f = tz + sv + rw,$$

where $t, s, r \ge 0$, $t + s + r = 1$, $z \in e_0 + K$, $v \in -e_0 + K$, $w \in B_{\ell_1}$.
Since $F_1(z) = F_1(e_0) = 1$, $F_1(v) = F_2(e_0) = -1$, $F_1(w) \le \|F_1\|_\infty \|w\|_1 \le 1$, we have

$$1 = F_1(f) = t - s + rF_1(w) \le t - s + r \le t + s + r = 1.$$

Thus the above inequalities are in fact equalities. This means that $s = 0$, and either $r = 0$ or $F_1(w) = 1$. If $F_1(w) = 1$, we necessarily have $w = \sum_{i \ge 0} \alpha_i e_i$ with $\alpha_i \ge 0$ $(i \ge 0)$, and if $r = 0$ we can take $w = 0$. In both cases, for each $i \ge 2$, we have

$$-2^{-i} b = f(u_i) = tz(u_i) + (1 - t)w(u_i) \ge -4^{-i} t \ge -4^{-i}.$$

It follows that $b \le 2^{-i}$ for each $i \ge 2$, and hence $b \le 0$, which is a contradiction that completes the proof. $\square$

Observation. Note that Claim 1 and the second part of Claim 2 imply that the line $F_1|_L = 1$ is tangent to the "half-sphere" $\{ae_0 + bg \in S_L : b \ge 0\}$ at e_0.

Second step of construction. For better understanding of the following geometric construction in L, the reader is invited to sketch a simple diagram.

The line $F_1|_L = 1$ supports B_L at e_0. Hence, if we consider an appropriate coordinate system, centered at e_0 and with axis of abscissae on the line $F_1|_L = 1$, then the points of S_L that are sufficiently near to e_0 will form the graph of a convex function, defined in a neighborhood of the origin of the axis of abscissae. By Observation above, we can apply Lemma 7.2 to get pairwise distinct points $f_n = a_n e_0 + b_n g \in S_L$ ($n \in \mathbb{N}$) such that $a_n, b_n > 0$, $b_n \searrow 0$, $a_n \to 1$, each line $\Lambda_n = \mathrm{aff}\{f_n, f_{n+1}\}$ is disjoint from B_L, and the angle between Λ_n and the line $F_1|_L = 1$ tends decreasingly to 0.

Observe that the lines Λ_1 and $u_0 = 1$ are not parallel since their angle is greater than the one between Λ_1 and $F_1|_L = 1$. Let $h \in L$ be the common point of the lines Λ_1 and $u_0|_L = -1$. By our construction, the compact convex set

$$C = \overline{\mathrm{conv}}\big[\{\pm f_j\}_{j\geq 2} \cup \{\pm h\}\big]$$

contains B_L, we have

$$\mathrm{ext}C = \{h, f_2, f_3, \ldots, e_0, -h, -f_2, -f_3, \ldots, -e_0\},$$

and $\partial_L C$ (the boundary of C in L) consists of the segments $[h, f_2]$, $[f_2, f_3]$, $[f_3, f_4], \ldots, [e_0, -h], [-h, -f_2], [-f_2, -f_3], [-f_3, -f_4], \ldots, [-e_0, h]$. Define

$$W = \overline{\mathrm{conv}}^{w^*}[V \cup C] = \mathrm{conv}[V \cup C].$$

Then W is the dual unit ball of an equivalent norm $|||\cdot|||$ on c_0, given by

$$|||x||| := \max x(W) = \max\big\{\|x\|, \max x(C)\big\}.$$

Denote $E = (c_0, |||\cdot|||)$.

Define $Y = L^\top$. Then Y is a subspace of codimension two in E, and $(E/Y)^* = Y^\perp = L$. Since, $B_{(L,|||\cdot|||)} = C$ is not a polytope, the quotient E/Y is not polyhedral.

Claim 3. *E is polyhedral.*

Proof. Notice that $W = \overline{\mathrm{conv}}^{w^*} B$, where

$$B = \{\pm e_i\}_{i\geq 0} \cup \big\{\pm e_0 \pm 4^{-i}(e_1 - e_i)\big\}_{i\geq 2} \cup \{\pm f_j\}_{j\geq 2} \cup \{\pm h\}. \tag{11}$$

Moreover, B is a boundary for E (since $f_j \to e_0$ and $e_0 \pm 4^{-i}(e_1 - e_i) \to e_0$), and the only w^*-limit points of B are the three points $0, \pm e_0$. Observe that $K^\top = \mathbb{R}u_0$. Thus E is polyhedral by Proposition 7.1. □

Claim 4. $Y^\perp \subset NA(E)$.

Proof. We have to show that, for each $f \in S_E \cap Y^\perp = \partial W \cap L = \partial_L C$, there exists a nonzero $x \in E$ such that $f(x) = |||x|||$ $(= \max x(W))$.

If $f \in [e_0, -h]$ or $f \in [-e_0, h]$, we can take $x = u_0$ or $x = -u_0$, respectively. If f belongs to any other of the segments that compose $\partial_L C$ (the boundary of C in L), then this segment is contained in one of the lines Λ_n. Moreover, this Λ_n is disjoint from V and supports C at f. Since V is w^*-compact and Λ_n is w^*-closed, the Hahn–Banach separation theorem gives existence of some $x \in E \setminus \{0\}$ such that $\max x(V) < \inf x(\Lambda_n) =: \alpha$. Since x is necessarily constant on Λ_n, we have $\max x(W) \leq \alpha = f(x)$. □

Claim 5. *Y is not proximinal in E.*

Proof. We want to show that $q(S_E) \neq S_{E/Y}$, where $q: E \to E/Y$ is the quotient map. Since (in canonical identifications) $L = (E/Y)^*$, we have $E/Y = (E/Y)^{**} = L^*$. Thus we can identify q with the restriction map

$$q: E \to L^*, \qquad x \mapsto x|_L. \tag{12}$$

We have $F_1|_L \in S_{L^*}$ since $\max F_1(C) = F_1(e_0) = 1$. Let us prove that $F_1|_L \notin q(S_E)$. If this is not the case, there exists $x \in S_E$ with $x|_L = F_1|_L$. In particular, $e_0(x) = F_1(e_0) = 1$. Since $\|e_0\| = |||e_0||| = 1$, the inclusion $B_{E^*} \supset B_{X^*}$ and Claim 1 imply that $x \in D_{E^*}(e_0) \subset D_{X^*}(e_0) = [F_1, F_2]$. But this implies that $x = u_0$ since $[F_1, F_2] \cap E = \{u_0\}$. Thus we get $F_1|_L = u_0|_L$, a contradiction since $F_1(g) \neq 0 = g(u_0)$. □

The proof of Example 7.3 is complete.

8. Third example

In this section we provide the following example which shows that, in the notation of Theorem 0.2, the implication (B) $\Rightarrow$ (D) does not hold for general polyhedral spaces. (We already know from Theorem 5.4 that it holds under the additional assumption that X satisfies (Δ).)

Example 8.1. There exists a polyhedral Banach space E, isomorphic to c_0, and a closed subspace $Y \subset E$ of codimension two, such that Y is proximinal and E/Y is not polyhedral.

The proof of Example 8.1 will go in a similar, but simpler, way as that of Example 7.3.

First step of construction. Let $\{u_i\}_{i\geq 0}$ and $\{e_i\}_{i\geq 0}$ be the canonical bases (indices starting from zero!) of c_0 and $\ell_1 = (c_0)^*$, respectively. Define

$$K = \overline{\mathrm{conv}}\big\{\pm\frac{1}{i}e_i : i \geq 1\big\},$$
$$V = \overline{\mathrm{conv}}^{w^*}\big[B_{\ell_1} \cup \pm(e_0 + K)\big] = \mathrm{conv}\big[B_{\ell_1} \cup \pm(e_0 + K)\big]. \tag{13}$$

Then V is the dual unit ball of an equivalent norm $\|\cdot\|$ on c_0, given by $\|x\| = \max x(V)$. We define $X = (c_0, \|\cdot\|)$.

Observe that $\mathrm{span}K \subset \mathrm{Ker}(u_0) \subset X^*$, but $\mathrm{span}K \neq \mathrm{Ker}(u_0)$ by the Baire category theorem (indeed, $\mathrm{span}K = \bigcup_{n\geq 1} nK$ while K has empty relative interior in $\mathrm{Ker}(u_0)$). Fix an arbitrary $g \in \mathrm{Ker}(u_0) \setminus \mathrm{span}K$ and define $L \subset X^*$ by

$$L = \mathrm{span}\{e_0, g\}.$$

Since u_0 attains its maximum over V at e_0, we have $e_0 \in S_{X^*}$.

Claim 1′. *$D_{X^*}(e_0) = \{u_0\}$. Consequently, $D_L(e_0) = \{u_0|_L\}$ by the Hahn–Banach theorem.*

Proof. If $F \in D_{X^*}(e_0)$ then $F|_K \equiv 0$ and $F(e_0) = 1$. Hence $F = u_0$. The other implication is obvious. □

Claim 2′. *If $f \in S_L$ and $f \neq e_0$, then $f(u_0) < 1$.*

Proof. If $f \in S_L$ and $f(u_0) = 1$, then (13) implies that $f \in e_0 + K$. On the other hand, $f = e_0 + bg$ for some $b \in \mathbb{R}$, since $f(u_0) = 1$ and $g(u_0) = 0$. Thus $bg \in K$, which is possible only if $b = 0$. □

Second step of construction. By Claim 1′, the line $u_0|_L = 1$ is tangent to S_L at e_0; and by Claim 2′, e_0 is the unique common point of this line and S_L. As in the "Second step of construction" in the proof of Example 7.3, we can apply Lemma 7.2 to get pairwise distinct points $f_n = a_n e_0 + b_n g \in S_L$ ($n \in \mathbb{N}$) such that $a_n, b_n > 0$, $b_n \searrow 0$, $a_n \to 1$, each line $\Lambda_n = \mathrm{aff}\{f_n, f_{n+1}\}$ is disjoint from B_L, and the angle between Λ_n and the line $u_0|_L = 1$ tends decreasingly to 0.

Let $h \in L$ be the common point of the lines Λ_1 and $u_0|_L = -1$. As in the proof of Example 7.3, the compact convex set

$$C = \overline{\mathrm{conv}}\big[\{\pm f_j\}_{j\geq 2} \cup \{\pm h\}\big]$$

contains B_L, its extreme points are the points $h, f_2, f_3, \ldots, e_0, -h - f_2, -f_3, \ldots, -e_0$, and its boundary (in L) consists of the segments $[h, f_2]$, $[f_2, f_3]$, $[f_3, f_4]$, $\ldots$, $[e_0, -h]$, $[-h, -f_2]$, $[-f_2, -f_3]$, $[-f_3, -f_4]$, $\ldots$, $[-e_0, h]$. Define

$$W = \overline{\mathrm{conv}}^{w^*}[V \cup C] = \mathrm{conv}[V \cup C].$$

Then W is the dual unit ball of an equivalent norm $|||\cdot|||$ on c_0, given by $|||x||| := \max x(W) = \max\big\{\|x\|, \max x(C)\big\}$. Denote $E = (c_0, |||\cdot|||)$.

Define $Y = L^{\top}$. Then Y is a subspace of codimension two in E, and $(E/Y)^* = Y^{\perp} = L$. Since, $B_{(L,|||\cdot|||)} = C$ is not a polytope, the quotient E/Y is not polyhedral.

Claim 3′. *E is polyhedral.*

Proof. The proof is identical to that of Claim 3 in the proof of Example 7.3. □

Claim 4′. *Y is proximinal in E.*

Proof. As in Claim 5 (proof of Example 7.3), we can canonically identify the quotient map $q: E \to X/E$ with the restriction map (12). We have to show that $q(S_E) = S_{L^*}$.

Let $\ell \in S_{L^*}$. There exists $f \in S_L = \partial_L C$ such that the line $\ell = 1$ supports C at f. If $f = e_0$, then $\ell = u_0|_L$ (Claim 1′), that is $\ell = q(u_0)$. Let $f \neq e_0$. Then the line $\ell = 1$ is disjoint from B_{X^*}. As in the proof of Claim 4 (proof of Example 7.3), the Hahn–Banach separation theorem (applied to the sets B_{X^*} and $\ell^{-1}(1)$ in the w^*-topology) gives a nonzero $x \in X$ such that $|||x||| = \sup x(W) = 1$ and $x|_L = \ell$. Then $x \in S_E$ and $\ell = q(x)$. □

The proof of Example 8.1 is complete.

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References

[1] D. Amir, F. Deutsch, Suns, moons, and quasi-polyhedra, J. Approx. Theory 6 (1972) 176–201.

[2] R. Durier, P.L. Papini, Polyhedral norms in an infinite-dimensional space, Rocky Mountain J. Math. 23 (1993) 863–875.

[3] S. Dutta, D. Narayana, Strongly proximinal subspaces of finite codimension in $C(K)$, Colloq. Math. 109 (2007) 119–128.
[4] V.P. Fonf, Polyhedral Banach spaces, Math. Notes USSR 30 (1981) 809–813.
[5] V.P. Fonf, On the boundary of a polyhedral Banach space, Extracta Math. 15 (2000) 145–154.
[6] V.P. Fonf, J. Lindenstrauss, Some results on infinite-dimensional convexity, Israel J. Math. 108 (1998) 13–32.
[7] V.P. Fonf, J. Lindenstrauss, On quotients of polyhedral spaces, preprint 1998, unpublished.
[8] V.P. Fonf, J. Lindenstrauss, On the metric projection in a polyhedral space, preprint 2003, unpublished.
[9] V.P. Fonf, L. Veselý, Infinite-dimensional polyhedrality, Canad. J. Math. 56 (2004) 472–494.
[10] G. Godefroy, V. Indumathi, Proximinality in subspaces of c_0, J. Approx. Theory 101 (1999) 175–181.
[11] G. Godefroy, V. Indumathi, Strong proximinality and polyhedral spaces, Rev. Mat. Complut. 14 (2001) 105–125.
[12] V. Indumathi, Metric projections of closed subspaces of c_0 onto subspaces of finite codimension, Colloq. Math. 99 (2004) 231–252.
[13] V. Indumathi, Metric projections and polyhedral spaces, Set-Valued Anal. 15 (2007) 239–250.
[14] V. Klee, Some characterizations of convex polyhedra, Acta Math. 102 (1959) 79–107.
[15] V. Klee, Polyhedral sections of convex bodies, Acta Math. 103 (1960) 243–267.
[16] J. Lindenstrauss, Notes on Klee's paper "Polyhedral sections of convex bodies", Israel J. Math. 4 (1966) 235–242.
[17] R.R. Phelps, Čebyšev subspaces of finite codimension in $C(X)$, Pacific J. Math. (13) (1963) 647–655.
[18] A.W. Roberts, D.E. Varberg, Convex Functions, in: Pure and Applied Mathematics, vol. 57, Academic Press, New York, London, 1973.
[19] L. Veselý, Generalized centers of finite sets in Banach spaces, Acta Math. Univ. Comenian. 66 (1997) 83–115.
[20] L. Veselý, Boundary of polyhedral spaces: an alternative proof, Extracta Math. 15 (2000) 213–217.
[21] L. Veselý, Metric projections in polyhedral spaces, preprint 2005, unpublished.